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Preprint Nr. 30/2009 — 04. November 2009

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Impressum:

Herausgeber:
Institut für Mathematik
Universität Augsburg
86135 Augsburg
http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml

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Upper and Lower Estimates for Invariance Entropy*†

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September 21, 2009

Abstract

Invariance entropy measures the minimal information rate necessary to render a subset of the state space of a continuous-time control system invariant. In the present paper, we derive upper and lower bounds for the invariance entropy of control systems on smooth manifolds, using differential-geometric tools. As an example, we compute these bounds explicitly for projected bilinear control systems on the unit sphere.

Keywords: Nonlinear Control Systems, Invariance Entropy, Bilinear Control Systems

1 Introduction

In [9], Nair, Evans, Mareels, and Moran introduced *topological feedback entropy* as a measure of the inherent rate at which a discrete-time control system generates stability information. They proved that the infimal data rate necessary to stabilize the control system into a compact subset of the state space is exactly given by that measure. For continuous-time systems on Euclidean space the notion of *invariance entropy* was established for the same purpose in [3]. Here, a connection to data rates can be found in the PhD thesis [8]. In the present paper, we show that the concept of invariance entropy can be extended naturally to control systems on arbitrary smooth manifolds. We further derive upper and lower bounds, which can be computed directly from the right-hand side of the system, and which generalize the estimates given in Theorem 4.1 and Theorem 4.2 of [3].

*This work was supported by DFG grant Co 124/17-1 within DFG priority program 1305.

†AMS Subject Classification 34C40, 93C15, 94A17

Consider a smooth manifold M, endowed with a metric d (not necessarily a Riemannian distance), and a control system

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in \mathcal{U},$$

on M with a smooth right-hand side $F : M \times \mathbb{R}^m \to TM$ and L^∞-controls taking values in a compact control range $U \subset \mathbb{R}^m$. Let the unique solution to the initial value problem $x(0) = x_0$ for the control function u be denoted by $\varphi(\cdot, x_0, u)$. Let $K, Q \subset M$ be compact sets with $K \subset Q$ and Q being controlled invariant. Then the invariance entropy $h_{\mathrm{inv}}(K, Q)$ is defined as follows: For each $T, \varepsilon > 0$ a set $\mathcal{S} \subset \mathcal{U}$ is called (T, ε, K, Q)-spanning set if for all $x \in K$ there is $u \in \mathcal{S}$ with $\inf_{y \in Q} d(\varphi(t, x, u), y) < \varepsilon$ for all $t \in [0, T]$. The minimal cardinality of such a set is denoted by $r_{\mathrm{inv}}(T, \varepsilon, K, Q)$ and

$$h_{\mathrm{inv}}(K, Q) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T, \varepsilon, K, Q).$$

It is easy to see that the limit in the definition above exists and that $h_{\mathrm{inv}}(K, Q)$ does not depend on the metric d.

The first main theorem of the present paper, Theorem 12, yields the following upper bound for $h_{\mathrm{inv}}(K, Q)$, depending on a Riemannian metric g imposed on M:

$$h_{\mathrm{inv}}(K, Q) \leq \max \left\{ 0, \max_{(x,u) \in Q \times U} \lambda_{\max}(S\nabla F_u(x)) \right\} \cdot \overline{\dim}_B(K). \tag{1}$$

Here $S\nabla F_u$ denotes the symmetrized covariant derivative of the vector field $F_u(\cdot) = F(\cdot, u)$, $\lambda_{\max}(\cdot)$ is the maximal eigenvalue, and $\overline{\dim}_B(K)$ the upper box dimension (or fractal dimension) of the set K. In order to obtain uniform Lipschitz constants on Q for the solution maps $\varphi(t, \cdot, u)$, the proof uses the *Wazewski Inequality*

$$\|D\varphi_{t,u}(x)\| \leq \exp\left(\int_0^t \lambda_{\max}(S\nabla F_{u(s)}(\varphi(s, x, u))) ds \right),$$

which serves as a substitute for the *Gronwall Lemma*, used in the proof of Theorem 4.2 in [3] (the Euclidean version of Theorem 12). Apart from that, the main arguments are similar. We like to note that an analogous inequality for the topological entropy of a flow φ on a smooth manifold M, induced by a differential equation $\dot{x} = f(x)$, was proved by A. Noack in her PhD thesis [10], namely

$$h_{\mathrm{top}}(\varphi|_K) \leq \max \left\{ 0, \max_{x \in K} \lambda_{\max}(S\nabla f(x)) \right\} \cdot \underline{\dim}_B(K),$$

where $K \subset M$ is a φ-invariant compact set, and $\underline{\dim}_B(K)$ denotes the lower box dimension of K. The proof of that inequality is primarily based on

an estimate of the topological entropy of maps (proved by Noack), which generalizes an earlier estimate of Ito [7]. Similar estimates for the topological entropy of a flow can be found in [1] and [2].
Our second main theorem, Theorem 14, yields a lower bound on $h_{\mathrm{inv}}(K,Q)$ depending on a volume form ω on M, namely

$$h_{\mathrm{inv}}(K,Q) \geq \max\left\{0, \min_{(x,u)\in Q\times U} \mathrm{div}_\omega F_u(x)\right\}, \tag{2}$$

where div_ω denotes the divergence with respect to ω. Here we need the additional assumption of K having positive volume. The proof is essentially based on the same arguments as the proof of Theorem 4.1 in [3], but uses a more general version of the *Liouville Formula*.
The present paper is organized as follows. In Section 2, we introduce notation and collect some facts on manifolds, upper box dimension and control systems. Section 5 introduces the concept of invariance entropy for control systems on smooth manifolds. Section 8 provides proofs of the Wazewski Inequality and the Liouville Formula. The main results, Theorem 12 and Theorem 14, and two corollaries are formulated and proved in Section 11. Finally, in Section 18, we compute the bounds (1) and (2) for projected bilinear systems on the unit sphere explicitly.

2 Notation and Preliminaries

2.1 Notation

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^d$, and $\mathbb{R}^{d\times d}$ we denote the standard sets. If (X,d) is a metric space, we write $\mathrm{cl}\,A$ for the topological closure of a set $A \subset X$. The ε-ball around $x \in X$ is denoted by $B_\varepsilon(x)$. The ε-neighborhood $N_\varepsilon(Q)$ of a set $Q \subset X$ is the union of all ε-balls centered at points in Q. By $\langle\cdot,\cdot\rangle$ we denote the standard Euclidean scalar product on $\mathbb{R}^d$. If F is a linear mapping between Euclidean spaces, $\|F\|$ denotes its operator norm, and F^* its adjoint (if the spaces have the same dimension). By $\lambda_{\max}(F)$ we denote the maximal eigenvalue of a self-adjoint endomorphism F. By I we denote the identity matrix. We write $\mathrm{Sym}(d,\mathbb{R})$ for the space of all real symmetric $d\times d$-matrices. The transposed of a matrix A is denoted by A^T, its trace by $\mathrm{tr}\,A$. We write $(v_1|\cdots|v_d)$ for the $d\times d$-matrix whose columns are $v_1,\ldots,v_d \in \mathbb{R}^d$. For any real number $r \in \mathbb{R}$ we let $\lfloor r\rfloor$ denote the integer part of r, i.e., the greatest integer less than or equal to r.
The term "smooth" always stands for C^∞. By a smooth manifold we mean a connected, finite-dimensional, second-countable, topological Hausdorff manifold endowed with a smooth differentiable structure. TM denotes the tangent bundle of the manifold M, T_xM is the tangent space at $x \in M$. For the derivative of a smooth mapping f (between manifolds) at the point x we write $Df(x)$. A diffeomorphism (between manifolds) is a smooth invertible

map with smooth inverse. The set of smooth vector fields on a manifold M is denoted by $\mathcal{X}(M)$. A Riemannian manifold (M, g) is a smooth manifold M endowed with a smooth Riemannian metric g. For the Levi-Civita connection associated with g we write ∇. A chart of a smooth d-dimensional manifold M is a pair (ϕ, V) such that $V \subset M$ is an open set and ϕ is a diffeomorphism from V onto an open subset of $\mathbb{R}^d$. The basis of T_xM, $x \in V$, associated with the chart (ϕ, V), is denoted by $\partial_1\phi_x, \ldots, \partial_d\phi_x$. If $\alpha : M \to \mathbb{R}$ is a smooth function, we write

$$\frac{\partial \alpha}{\partial \phi^i}(x) := \partial_i(\alpha \circ \phi^{-1})(\phi(x)),$$

where ∂_i is the partial derivative by the i-th argument. For the components of a Riemannian metric g and for the associated Christoffel symbols we use the standard notations, g_{ij} and Γ^k_{ij}. As usual, the components of the inverse of (g_{ij}) are denoted by g^{ij}. If $f \in \mathcal{X}(M)$, we write $\mathcal{L}_f$ for the Lie derivative along f. If $\varphi : M \to N$ is a diffeomorphism and ω is a volume form on N, we write $\varphi^*\omega$ for the pullback of ω via φ, i.e., $(\varphi^*\omega)(x)(v_1, \ldots, v_d) = \omega(\varphi(x))(D\varphi(x)v_1, \ldots, D\varphi(x)v_d)$ for all $x \in M$ and $v_1, \ldots, v_d \in T_xM$. In local formulas we do not use Einstein summation convention, but we omit the range of the indices, which always run from 1 to d, the dimension of the manifold.

2.2 Manifolds

Let (M, g) be a Riemannian manifold of dimension d and let $f \in \mathcal{X}(M)$. Then the *covariant derivative* $\nabla f(x)$ of f at $x \in M$ is a linear endomorphism of the tangent space T_xM, locally—with respect to a chart (ϕ, V)—given by

$$\nabla f(\cdot)v = \nabla_v f(\cdot) = \sum_{i,k} \left[\frac{\partial f^k}{\partial \phi^i} + \sum_j \Gamma^k_{ij} f^j \right] v^i \partial_k \phi. \tag{3}$$

The *symmetrized covariant derivative* of f at x is the self-adjoint endomorphism $S\nabla f(x) := \frac{1}{2}[\nabla f(x) + \nabla f(x)^*]$. In local coordinates, we can view $S\nabla f(\cdot)$ as a matrix $(s_{\mu\nu}(\cdot))$ whose entries satisfy

$$2s_{\mu\nu} = \frac{\partial f^\mu}{\partial \phi^\nu} + \sum_{\theta,\kappa} g^{\mu\theta} \frac{\partial f^\kappa}{\partial \phi^\theta} g_{\kappa\nu} + \sum_{i,l} f^i g^{\mu l} \frac{\partial g_{\nu l}}{\partial \phi^i}. \tag{4}$$

Let (M, ω) be a volume manifold, i.e., M is an orientable smooth manifold and ω is a smooth volume form on M. Then for a smooth map $\varphi : M \to M$ the determinant of $D\varphi(x) : T_xM \to T_{\varphi(x)}M$ with respect to ω is defined by

$$(\varphi^*\omega)(x) = \det{}_\omega D\varphi(x) \cdot \omega(x).$$

The divergence of $f \in \mathcal{X}(M)$ at x is defined by the equation

$$(\mathcal{L}_f \omega)(x) = \operatorname{div}_\omega f(x) \cdot \omega(x).$$

If $\alpha : M \to \mathbb{R}$ is a smooth and nowhere vanishing function, then also $\alpha \cdot \omega$ is a volume form on M and

$$\operatorname{div}_{\alpha\cdot\omega} f = \operatorname{div}_\omega f + \frac{\mathcal{L}_f \alpha}{\alpha}. \tag{5}$$

The Borel measure on M, induced by ω, is denoted by μ_ω. Let $\varphi : M \to \mathbb{R}$ be an integrable function with respect to the integral induced by μ_ω, and let $g : M \to M$ be a diffeomorphism. Then the transformation rule holds:

$$\int_{g(A)} \varphi(x) d\mu_\omega(x) = \int_A \varphi(g(y)) \cdot |\det_\omega Dg(y)| d\mu_\omega(y). \tag{6}$$

2.3 Upper Box Dimension

Next, we recall the definition of *upper box dimension* (cf. [1, Def. 2.2.1]): For a totally bounded subset Z of a metric space the minimal number of ε-balls needed to cover Z is denoted by $N(\varepsilon, Z)$, and the upper box dimension (or fractal dimension) of Z is given by

$$\overline{\dim}_B(Z) := \limsup_{\varepsilon \searrow 0} \frac{\ln N(\varepsilon, Z)}{\ln(1/\varepsilon)}.$$

The upper box dimension of a compact subset Z of a d-dimensional Riemannian manifold is at most d and if Z has nonvoid interior, it equals d. The following lemma shows that the upper box dimension of a set Z does not depend on the space it is embedded in.

Lemma 3 *Let (X, d) be a metric space and $Z \subset X$ a totally bounded set. Let $\overline{\dim}_B(Z; X)$ denote the upper box dimension of Z as a subspace of (X, d) and $\overline{\dim}_B(Z; Z)$ the upper box dimension of Z as a subspace of (Z, d). Then $\overline{\dim}_B(Z; X) = \overline{\dim}_B(Z; Z)$.*

Proof: By $N(\varepsilon, Z; X)$ ($N(\varepsilon, Z; Z)$) we denote the minimal cardinality of a covering of Z with ε-balls in X (in Z). For given $\varepsilon > 0$ let $\mathcal{B} = \{B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)\}$, $x_i \in X$, be a minimal covering of Z with ε-balls in X, i.e., in particular $n = N(\varepsilon, Z; X)$. Then for every $i \in \{1, \ldots, n\}$ there exists some $z_i \in B_\varepsilon(x_i) \cap Z$, since otherwise $\mathcal{B}$ would not be minimal. Let $\widetilde{\mathcal{B}} := \{B_{2\varepsilon}(z_1), \ldots, B_{2\varepsilon}(z_n)\}$. Now take an arbitrary point $z \in Z$. Then there exists $i \in \{1, \ldots, n\}$ with $d(z, x_i) < \varepsilon$. It follows that

$$d(z, z_i) \leq d(z, x_i) + d(x_i, z_i) < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, $\widetilde{\mathcal{B}}$ is a covering of Z consisting of n balls in Z of radius 2ε. This implies

$$N(2\varepsilon, Z; X) \leq N(2\varepsilon, Z; Z) \leq N(\varepsilon, Z; X).$$

Hence, for all $\varepsilon \in (0,1)$ it holds that

$$\frac{\ln N(2\varepsilon, Z; X)}{\ln(1/\varepsilon)} \leq \frac{\ln N(2\varepsilon, Z; Z)}{\ln(1/\varepsilon)} \leq \frac{\ln N(\varepsilon, Z; X)}{\ln(1/\varepsilon)}.$$

Using that $\ln(1/\varepsilon) = \ln(2) + \ln(1/(2\varepsilon))$ we obtain

$$\limsup_{\varepsilon \searrow 0} \frac{\ln N(2\varepsilon, Z; X)}{\ln(2) + \ln(1/(2\varepsilon))} \leq \limsup_{\varepsilon \searrow 0} \frac{\ln N(2\varepsilon, Z; Z)}{\ln(2) + \ln(1/(2\varepsilon))} \leq \overline{\dim}_B(Z; X).$$

Since

$$\frac{\ln N(2\varepsilon, Z; X)}{\ln(2) + \ln(1/(2\varepsilon))} = \underbrace{\frac{\ln(1/(2\varepsilon))}{\ln(2) + \ln(1/(2\varepsilon))}}_{\to 1 \text{ for } \varepsilon \to 0} \cdot \frac{\ln N(2\varepsilon, Z; X)}{\ln(1/(2\varepsilon))},$$

we obtain $\overline{\dim}_B(Z; X) \leq \overline{\dim}_B(Z; Z) \leq \overline{\dim}_B(Z; X)$. □

3.1 Control Systems

Let M be a d-dimensional smooth manifold. By a control system on M we understand a family

$$\dot{x}(t) = F(x(t), u(t)), \quad u \in \mathcal{U}, \tag{7}$$

of ordinary differential equations, with a right-hand side $F : M \times \mathbb{R}^m \to TM$ satisfying $F_u := F(\cdot, u) \in \mathcal{X}(M)$ for all $u \in \mathbb{R}^m$. For simplicity, we assume that F is smooth. (Indeed, for our purposes it would be sufficient to assume that F is continuous and each local representation of F is of class C^1 it its first variable). The family $\mathcal{U}$ of admissible control functions is given by

$$\mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \ : \ u \text{ measurable and } u(t) \in U \text{ a.e.}\}$$

with a compact control range $U \subset \mathbb{R}^m$. Smoothness of F in the first argument guarantees that for each control function $u \in \mathcal{U}$ and each initial value $x \in M$ there exists a unique solution $\varphi(\cdot, x, u)$ satisfying $\varphi(0, x, u) = x$, defined on an open interval containing $t = 0$. Note that in general $\varphi(\cdot, x, u)$ is only a solution in the sense of Carathéodory, i.e., a locally absolutely continuous curve satisfying the corresponding differential equation almost everywhere. (A curve $c : I \to M$ is locally absolutely continuous iff $\alpha \circ c$ is locally absolutely continuous in the usual sense for every smooth function $\alpha : M \to \mathbb{R}$.) We assume that all such solutions can be extended to the whole real line. In fact, for the purpose of studying invariance entropy, we may assume this without loss of generality, since we only consider solutions which do not leave a small neighborhood of a compact set. Hence, we obtain a mapping

$$\varphi : \mathbb{R} \times M \times \mathcal{U} \to M, \quad (t,x,u) \mapsto \varphi(t,x,u),$$

satisfying the *cocycle property*

$$\varphi(s, \varphi(t,x,u), \Theta_t u) = \varphi(s+t,x,u) \tag{8}$$

for all $t,s \in \mathbb{R}$, $x \in M$, $u \in \mathcal{U}$, where $(\Theta_t)_{t\in\mathbb{R}}$ denotes the shift flow on $\mathcal{U}$, defined by

$$(\Theta_t u)(s) \equiv u(t+s).$$

Instead of $\varphi(t,x,u)$ we also write $\varphi_{t,u}(x)$. Note that smoothness of the right-hand side F implies smoothness of $\varphi_{t,u}(\cdot)$.
Finally, we state a result on the approximation of arbitrary solutions by solutions corresponding to piecewise constant control functions, which easily follows from the combination of [6, Theo. 2.20] and [6, Theo. 2.24].

Proposition 4 *Consider control system* (7)*, let* $(x_0,u_0) \in M \times \mathcal{U}$ *and* $T > 0$*. Then for every* $\varepsilon > 0$ *there exist* $\delta > 0$ *and a piecewise constant control function* $u \in \mathcal{U}$ *such that* $d(x,x_0) < \delta$ *implies*

$$d\left(\varphi(t,x,u), \varphi(t,x_0,u_0)\right) \leq \varepsilon \quad \text{for all } t \in [0,T].$$

5 Invariance Entropy

Consider control system (7), and let d be a metric on M compatible with the given topology. Let $K, Q \subset M$ be compact sets with $K \subset Q$, and assume that Q is *controlled invariant*, i.e., for every $x \in Q$ there is $u \in \mathcal{U}$ such that $\varphi(t,x,u) \in Q$ for all $t \geq 0$. For given $T, \varepsilon > 0$ a set $\mathcal{S} \subset \mathcal{U}$ of control functions is called (T,ε,K,Q)*-spanning* if for all $x \in K$ there exists $u \in \mathcal{S}$ with $\varphi(t,x,u) \in N_\varepsilon(Q)$ for all $t \in [0,T]$. The minimal cardinality of such a set is denoted by $r_{\mathrm{inv}}(T,\varepsilon,K,Q)$, and the invariance entropy $h_{\mathrm{inv}}(K,Q)$ is defined as follows:

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon,K,Q) &:= \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T,\varepsilon,K,Q), \\ h_{\mathrm{inv}}(K,Q) &:= \lim_{\varepsilon \searrow 0} h_{\mathrm{inv}}(\varepsilon,K,Q). \end{aligned}$$

The arguments, given in [3], which show finiteness of $r_{\mathrm{inv}}(T,\varepsilon,K,Q)$ and existence of the limit in the definition above, naturally apply also to systems on manifolds; hence, we will not repeat them here. Next, we recall the definition of *strong invariance entropy*, introduced in [3] as an auxiliary quantity, which upper bounds $h_{\mathrm{inv}}(K,Q)$: Define the *lift* $\mathcal{Q}$ of Q by

$$\mathcal{Q} := \left\{(x,u) \in M \times \mathcal{U} \ : \ \varphi(t,x,u) \in Q \text{ for all } t \geq 0\right\}.$$

A subset $\mathcal{S}^+ \subset \mathcal{Q}$ is called *strongly* (T, ε, K, Q)*-spanning* if for every $x \in K$ there is $(y, v) \in \mathcal{S}^+$ with

$$d(\varphi(t, x, v), \varphi(t, y, v)) < \varepsilon \quad \text{for all } t \in [0, T].$$

By $r^+_{\mathrm{inv}}(T, \varepsilon, K, Q)$ we denote the minimal cardinality of such a set, and we define the strong invariance entropy $h^+_{\mathrm{inv}}(K, Q)$ by

$$\begin{aligned} h^+_{\mathrm{inv}}(\varepsilon, K, Q) &:= \limsup_{T \to \infty} \frac{1}{T} \ln r^+_{\mathrm{inv}}(T, \varepsilon, K, Q), \\ h^+_{\mathrm{inv}}(K, Q) &:= \lim_{\varepsilon \searrow 0} h^+_{\mathrm{inv}}(\varepsilon, K, Q). \end{aligned}$$

It is easy to see that $r_{\mathrm{inv}}(T, \varepsilon, K, Q) \le r^+_{\mathrm{inv}}(T, \varepsilon, K, Q)$ and hence

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon, K, Q) &\le h^+_{\mathrm{inv}}(\varepsilon, K, Q), \\ h_{\mathrm{inv}}(K, Q) &\le h^+_{\mathrm{inv}}(K, Q). \end{aligned} \tag{9}$$

For a proof see [3, Prop. 3.2] or [8, Prop. 3.1.3]. We also write $h_{\mathrm{inv}}(K, Q; F)$ or $h^+_{\mathrm{inv}}(K, Q; F)$ in order to refer to the system with right-hand side F, if there are different control systems in consideration.
The following proposition shows that both $h_{\mathrm{inv}}(K, Q)$ and $h^+_{\mathrm{inv}}(K, Q)$ are independent of the metric imposed on M.

Proposition 6 *$h_{\mathrm{inv}}(K, Q)$ and $h^+_{\mathrm{inv}}(K, Q)$ do not depend on the metric.*

Proof: Let d' be another metric on M inducing the given topology. Compactness of Q implies uniform continuity of the identity $\mathrm{id} : (M, d) \to (M, d')$ on Q, i.e.,

$$\forall \varepsilon > 0 : \ \exists \delta > 0 : \ \forall x \in Q : \ \forall y \in M : \ d(x, y) < \delta \Rightarrow d'(x, y) < \varepsilon.$$

Hence, every (T, δ, K, Q)-spanning set with respect to d is (T, ε, K, Q)-spanning with respect to d' if $\delta = \delta(\varepsilon)$ is chosen as above, and the same is true for strongly spanning sets. This implies the assertion. $\square$
The next proposition can be found as Proposition 3.4(iv) in [3] for systems on Euclidean space. It is clear that the proof also applies to systems on smooth manifolds and hence we omit it.

Proposition 7 *Consider the control systems* (7) *and*

$$\dot{x}(t) = s \cdot F(x(t), u(t)), \quad u \in \mathcal{U}, \tag{10}$$

where $s > 0$. Let $K, Q \subset M$ be compact sets with $K \subset Q$ and Q being controlled invariant with respect to system (7)*. Then Q is also controlled invariant with respect to system* (10) *and it holds that*

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon, K, Q; sF) &= s \cdot h_{\mathrm{inv}}(\varepsilon, K, Q; F) \text{ for all } \varepsilon > 0, \\ h_{\mathrm{inv}}(K, Q; sF) &= s \cdot h_{\mathrm{inv}}(K, Q; F). \end{aligned}$$

8 The Wazewski Inequality and the Liouville Formula

In this section, we provide proofs for the Wazewski Inequality and the Liouville Formula. In the first proof, we will use the well-known formula

$$\frac{d}{dt} g_{x(t)} \left(X(t), Y(t)\right) = g_{x(t)} \left(\frac{DX}{dt}(t), Y(t)\right) + g_{x(t)} \left(X(t), \frac{DY}{dt}(t)\right) \tag{11}$$

which holds for vector fields $X, Y : I \to TM$ along a smooth curve $x : I \to M$ on a Riemannian manifold (M, g), where $\frac{D}{dt}$ denotes the covariant derivative along x. By an elementary computation in local coordinates it can be proved that this formula holds almost everywhere on I if x, X and Y are only locally absolutely continuous.

Proposition 9 *Consider control system* (7) *and let g be a smooth Riemannian metric on M.*

(i) *For arbitrary $(x, u) \in M \times \mathcal{U}$ and $v \in T_xM$ the curve*

$$c_{x,u,v} : t \mapsto D\varphi_{t,u}(x)v, \quad c_{x,u,v} : \mathbb{R} \to TM,$$

is locally absolutely continuous and satisfies the Riemannian variational equation

$$\frac{Dz}{dt}(t) = \nabla F_{u(t)}(\varphi_{t,u}(x))z(t) \tag{12}$$

almost everywhere, where $\frac{D}{dt}$ denotes the covariant derivative along the solution $\varphi(\cdot, x, u)$.

(ii) *For all $t \geq 0$ the inequality*

$$\|D\varphi_{t,u}(x)\| \leq \exp\left(\int_0^t \lambda_{\max}(S\nabla F_{u(s)}(\varphi_{s,u}(x)))ds\right)$$

holds.

Proof:

(i) We abbreviate $c_{x,u,v}$ by c and $\varphi_{t,u}(x)$ by x_t. Let the local expressions of x_t, $F_{u(t)}$ and $c(t)$ with respect to a chart (ϕ, V) be

$$\begin{aligned} \phi(x_t) &= (x^1(t), \ldots, x^d(t)), \quad F_{u(t)}(x) = \sum_i F_t^i(x) \partial_i \phi_x, \\ c(t) &= \sum_i c^i(t) \partial_i \phi_{x_t}. \end{aligned}$$

By (3), the local expression of $\nabla F_{u(t)}(x)$ is given by

$$\nabla F_{u(t)}(x)w = \sum_{i,j} \frac{\partial F_t^i}{\partial \phi^j}(x) w^j \partial_i \phi_x + \sum_{i,j,k} \Gamma_{ij}^k(x) F_t^i(x) w^j \partial_k \phi_x.$$

From the variational equation for Carathéodory differential equations in Euclidean space it follows that c is locally absolutely continuous with

$$\dot{c}^i(t) = \sum_j \frac{\partial F_t^i}{\partial \phi^j}(x_t)c^j(t) \quad \text{a.e.,} \;\; i = 1, \ldots, d.$$

Hence, the right-hand side of (12) (with $z(t) = c(t)$) is (almost everywhere) given by

$$\begin{aligned} &\sum_{i,j} \frac{\partial F_t^i}{\partial \phi^j}(x_t)c^j(t)\partial_i \phi_{x_t} \; + \sum_{i,j,k} \Gamma_{ij}^k(x_t)F_t^i(x_t)c^j(t)\partial_k\phi_{x_t} \\ &\qquad\qquad\qquad = \dot{c}(t) + \sum_{i,j,k} \Gamma_{ij}^k(x_t)\dot{x}^i(t)c^j(t)\partial_k\phi_{x_t}. \end{aligned}$$

For the left-hand side we obtain

$$\begin{aligned} \frac{Dc}{dt}(t) \;&=\; \frac{D}{dt}\left[\sum_j c^j(t)\partial_j\phi_{x_t}\right] = \sum_j \left[\dot{c}^j(t)\partial_j\phi_{x_t} + c^j(t)\frac{D\partial_j\phi_{x_t}}{dt}(t)\right] \\ &= \dot{c}(t) + \sum_j c^j(t)\left(\nabla_{\dot{x}_t}\partial_j\phi\right)(x_t) \\ &= \dot{c}(t) + \sum_j c^j(t)\left(\nabla_{\sum_i \dot{x}^i(t)\partial_i\phi_{x_t}}\partial_j\phi\right)(x_t) \\ &= \dot{c}(t) + \sum_{i,j} \dot{x}^i(t)c^j(t)\left(\nabla_{\partial_i\phi_{x_t}}\partial_j\phi\right)(x_t) \\ &= \dot{c}(t) + \sum_{i,j,k} \Gamma_{ij}^k(x_t)\dot{x}^i(t)c^j(t)\partial_k\phi_{x_t}. \end{aligned}$$

This proves assertion (i).

(ii) Let $x_t :\equiv \varphi_{t,u}(x)$ and $\lambda(t) :\equiv \lambda_{\max}(S\nabla F_{u(t)}(\varphi_{t,u}(x)))$. Let $z : \mathbb{R} \to TM$ be a locally absolutely continuous solution of the variational equation (12). Then for almost all $t \in \mathbb{R}$ we obtain

$$\begin{aligned} \frac{d}{dt}\|z(t)\|^2 \;&=\; \frac{d}{dt}g_{x_t}(z(t), z(t)) \\ &\overset{(11)}{=} g_{x_t}\left(\frac{Dz}{dt}(t), z(t)\right) + g_{x_t}\left(z(t), \frac{Dz}{dt}(t)\right) \\ &= g_{x_t}\left(\nabla F_{u(t)}(x_t)z(t), z(t)\right) + g_{x_t}\left(z(t), \nabla F_{u(t)}(x_t)z(t)\right) \\ &= g_{x_t}\left(\nabla F_{u(t)}(x_t)z(t), z(t)\right) + g_{x_t}\left(\nabla F_{u(t)}(x_t)^* z(t), z(t)\right) \\ &= 2g_{x_t}\left(\tfrac{1}{2}\left[\nabla F_{u(t)}(x_t) + \nabla F_{u(t)}(x_t)^*\right]z(t), z(t)\right) \\ &\leq 2\lambda(t)\|z(t)\|^2. \end{aligned}$$

Now we assume that $z(t) \neq 0$ for all $t \geq 0$. This implies for almost all $t \geq 0$

$$\begin{aligned}
\frac{\frac{d}{dt}\|z(t)\|^2}{\|z(t)\|^2} \leq 2\lambda(t) \quad &\Rightarrow \quad \int_0^t \frac{\frac{d}{ds}\|z(s)\|^2}{\|z(s)\|^2} ds \leq 2\int_0^t \lambda(s)ds \\
&\Rightarrow \quad \ln\left(\|z(t)\|^2\right) - \ln\left(\|z(0)\|^2\right) \leq 2\int_0^t \lambda(s)ds \\
&\Rightarrow \quad \ln\|z(t)\| - \ln\|z(0)\| \leq \int_0^t \lambda(s)ds \\
&\Rightarrow \quad \|z(t)\| \leq \|z(0)\| \exp\left(\int_0^t \lambda(s)ds\right).
\end{aligned}$$

In order to show that λ is locally integrable (and hence the integral above exists) let (ϕ, V) be a chart such that $\varphi(I, x, u) \subset V$ for some open interval I. Then $\lambda = \lambda_{\max} \circ A$ on I, where $A : I \to \mathrm{Sym}(d, \mathbb{R})$ is given by (see (4))

$$\begin{aligned}
2\left[A(t)\right]_{\mu\nu} = \; & \frac{\partial F^{\mu}_{u(t)}}{\partial \phi^{\nu}}(x_t) + \sum_{\theta,\kappa} g^{\mu\theta}(x_t)\frac{\partial F^{\kappa}_{u(t)}}{\partial \phi^{\theta}}(x_t)g_{\kappa\nu}(x_t) \\
& + \sum_{i,l} F^i_{u(t)}(x_t)g^{\mu l}(x_t)\frac{\partial g_{\nu l}}{\partial \phi^i}(x_t).
\end{aligned}$$

The function $\lambda_{\max}$ is continuous, since eigenvalues depend continuously on the matrix. A is measurable, since both $F^i_{u(t)}(x_t)$ and $\frac{\partial F^i_{u(t)}}{\partial \phi^j}(x_t)$ depend measurably on t, which follows from the facts that F is continuously differentiable (in the first argument), x_t is continuous and u is measurable. Finiteness of the integral (over compact time intervals) follows from compactness of the control range U.
Since for each $v \in T_xM\backslash\{0\}$ the function $z(t) = D\varphi_{t,u}(x)v$ is a solution of (12) with $z(t) \neq 0$ for all $t \geq 0$, we obtain

$$\begin{aligned}
\|D\varphi_{t,u}(x)\| &= \max_{\|v\|=1} \|D\varphi_{t,u}(x)v\| \\
&\leq \max_{\|v\|=1} \|\underbrace{D\varphi_{0,u}(x)}_{=\mathrm{id}} v\| \exp\left(\int_0^t \lambda(s)ds\right) \\
&= \exp\left(\int_0^t \lambda(s)ds\right).
\end{aligned}$$

This finishes the proof of (ii).

□

For the proof of our second main result we need the following version of the Liouville Formula:

Proposition 10 *Consider control system (7) and let ω be a smooth volume form on M. Then for all $(t,x,u) \in \mathbb{R}_0^+ \times M \times \mathcal{U}$ it holds that*

$$\det\nolimits_\omega D\varphi_{t,u}(x) = \exp\left(\int_0^t \operatorname{div}_\omega F_{u(s)}(\varphi_{s,u}(x))ds\right). \tag{13}$$

Proof: We fix $(x,u) \in M \times \mathcal{U}$. For brevity we write $X_t = F_{u(t)}$ and $x_t = \varphi_{t,u}(x)$ for all $t \in \mathbb{R}$. First we prove that the following identity holds:

$$\frac{d}{dt}\varphi_{t,u}^*\omega = \varphi_{t,u}^*(\mathcal{L}_{X_t}\omega) \text{ for almost all } t \in \mathbb{R}. \tag{14}$$

It suffices to prove formula (14) locally (in $\mathbb{R}^d$). Then we have $\omega = \alpha \cdot \omega_0$ with the standard volume form $\omega_0 = dx^1 \wedge \cdots \wedge dx^d$ and a smooth function $\alpha : \mathbb{R}^d \to \mathbb{R}$. Let $v_1, \ldots, v_d \in \mathbb{R}^d$ be vectors such that (without loss of generality) $\det(v_1|\cdots|v_d) = 1$. Then for all $t \in \mathbb{R}$ we obtain

$$\begin{aligned}
\varphi_{t,u}^*\omega(x)(v_1,\ldots,v_d) &= \alpha(x_t)\det(D\varphi_{t,u}(x)v_1|\cdots|D\varphi_{t,u}(x)v_d)\\
&= \alpha(x_t)\det\left[D\varphi_{t,u}(x)\cdot(v_1|\cdots|v_d)\right]\\
&= \alpha(x_t)\det D\varphi_{t,u}(x).
\end{aligned}$$

For almost all $t \in \mathbb{R}$ the derivatives $\frac{d}{dt}\varphi_{t,u}(x) = \dot{x}_t$ and $\frac{d}{dt}D\varphi_{t,u}(x)$ exist. For those t-values we have

$$\begin{aligned}
\frac{d}{dt}\varphi_{t,u}^*\omega(x)(v_1,\ldots,v_d) &= \frac{d}{dt}\left(\alpha(x_t)\det D\varphi_{t,u}(x)\right)\\
&= \langle\nabla\alpha(x_t),\dot{x}_t\rangle\det D\varphi_{t,u}(x) + \alpha(x_t)\frac{d}{dt}\det D\varphi_{t,u}(x).
\end{aligned}$$

By the usual Liouville Formula for Carathéodory differential equations on Euclidean space we have

$$\frac{d}{dt}\det D\varphi_{t,u}(x) = \operatorname{tr} DX_t(x_t)\det D\varphi_{t,u}(x).$$

This leads to

$$\begin{aligned}
\frac{d}{dt}\varphi_{t,u}^*\omega(x)(v_1,\ldots,v_d) &= \langle\nabla\alpha(x_t),X_t(x_t)\rangle\det D\varphi_{t,u}(x)\\
&\quad+\ \alpha(x_t)\operatorname{tr} DX_t(x_t)\det D\varphi_{t,u}(x)\\
&= \left(\langle\nabla\alpha,X_t\rangle + \alpha\operatorname{tr} DX_t\right)(x_t)\det D\varphi_{t,u}(x).
\end{aligned}$$

For the right-hand side of (14) we obtain

$$\begin{aligned}
\varphi_{t,u}^*(\mathcal{L}_{X_t}\omega)(x)(v_1,\ldots,v_d) &= \varphi_{t,u}^*(\operatorname{div}_\omega X_t\cdot\omega)(x)(v_1,\ldots,v_d)\\
&\overset{(5)}{=} \varphi_{t,u}^*\left(\left(\alpha\operatorname{div}_{\omega_0} X_t + \langle\nabla\alpha,X_t\rangle\right)\omega_0\right)(x)(v_1,\ldots,v_d)\\
&= \left(\alpha\operatorname{tr} DX_t + \langle\nabla\alpha,X_t\rangle\right)(x_t)\det D\varphi_{t,u}(x).
\end{aligned}$$

This proves (14). In order to show the assertion, we have to prove that

$$\ln \det_\omega D\varphi_{t,u}(x) = \int_0^t \operatorname{div}_\omega X_s(x_s)ds \quad \text{for all } t \geq 0. \tag{15}$$

Note that the integral on the right-hand side of the equation exists, since the function

$$t \mapsto \operatorname{div}_\omega X_t(x_t) = \operatorname{div}_\omega F_{u(t)}(\varphi_{t,u}(x))$$

is the composition of the measurable function $t \mapsto (\varphi(t,x,u), u(t))$, $\mathbb{R} \to M \times \mathbb{R}^m$, and the continuous function $(p,v) \mapsto \operatorname{div}_\omega F_v(p)$, $M \times \mathbb{R}^m \to \mathbb{R}$, and it is essentially bounded on compact intervals: For almost all $s \in [0,t]$ one has

$$\left|\operatorname{div}_\omega F_{u(s)}(\varphi_{s,u}(x))\right| \leq \max_{(z,v)\in\varphi([0,t],x,u)\times U} \left|\operatorname{div}_\omega F_v(z)\right|.$$

For $t = 0$ both sides of equation (15) coincide, since $\varphi_{0,u} = \mathrm{id}_M$ and hence $\det_\omega D\varphi_{0,u}(x) \equiv 1$. Therefore, it suffices to show that the derivatives of both sides coincide almost everywhere:

$$\begin{aligned}
\tfrac{d}{dt} \ln \det_\omega D\varphi_{t,u}(x) &= (\det_\omega D\varphi_{t,u}(x))^{-1} \frac{d}{dt} \det_\omega D\varphi_{t,u}(x) \\
&= (\det_\omega D\varphi_{t,u}(x))^{-1} \frac{d}{dt} \frac{\varphi_{t,u}^*\omega(x)}{\omega(x)} \\
&\overset{(14)}{=} (\det_\omega D\varphi_{t,u}(x))^{-1} \frac{\varphi_{t,u}^*(\mathcal{L}_{X_t}\omega)(x)}{\omega(x)} \\
&= (\det_\omega D\varphi_{t,u}(x))^{-1} \frac{\varphi_{t,u}^*([\operatorname{div}_\omega X_t] \cdot \omega)(x)}{\omega(x)} \\
&= \frac{\omega(x)}{(\varphi_{t,u}^*\omega)(x)} \frac{\varphi_{t,u}^*([\operatorname{div}_\omega X_t] \cdot \omega)(x)}{\omega(x)} \\
&= \frac{\varphi_{t,u}^*([\operatorname{div}_\omega X_t] \cdot \omega)(x)}{(\varphi_{t,u}^*\omega)(x)} \\
&= \frac{\operatorname{div}_\omega X_t(x_t)}{\omega(x_t)} \omega(x_t) = \operatorname{div}_\omega X_t(x_t).
\end{aligned}$$

This implies the assertion. □

11 The Main Results

Now, we formulate and prove our main theorems. The first one yields an upper bound for the invariance entropy in terms of the symmetrized covariant derivative of the right-hand side vector fields of the given control system and the upper box dimension of the set K:

Theorem 12 *Consider control system* (7) *and let* $K, Q \subset M$ *be compact sets with* $K \subset Q$ *and* Q *being controlled invariant. Let* g *be a smooth Riemannian metric on* M*. Then the following estimate holds:*

$$h_{\mathrm{inv}}(K,Q) \leq \max\left\{0, \max_{(x,u)\in Q\times U} \lambda_{\max}(S\nabla F_u(x))\right\} \cdot \overline{\dim}_B(K).$$

Proof: The proof is subdivided into three parts.
<u>*Step 1:*</u> Let $\varepsilon > 0$ be chosen arbitrarily but small enough such that $\mathrm{cl}\, N_{2\varepsilon}(Q)$ is compact and for all $x \in Q$ the Riemannian exponential function $\exp_x$ is defined on $B_\varepsilon(0) \subset T_xM$. By compactness of Q and local compactness of M the first is possible; for the second see [5, Cor. 2.89]. Choose a smooth cut-off function $\theta : M \to [0,1]$ such that

$$\theta(x) \equiv 1 \text{ on } \mathrm{cl}\, N_\varepsilon(Q) \text{ and } \theta(x) \equiv 0 \text{ on } M\backslash N_{2\varepsilon}(Q).$$

We define a smooth mapping $\tilde{F} : M \times \mathbb{R}^m \to TM$ by

$$\tilde{F}(x,u) := \theta(x)F(x,u) \text{ for all } (x,u) \in M \times \mathbb{R}^m,$$

which serves as a new right-hand side:

$$\dot{x}(t) = \tilde{F}(x(t),u(t)), \quad u \in \mathcal{U}. \tag{16}$$

The corresponding solutions are denoted by $\tilde{\varphi}(t,x,u)$. By definition of $\tilde{F}$ we have

$$\varphi(t,x,u) = \tilde{\varphi}(t,x,u) \text{ whenever } \varphi([0,t],x,u) \subset \mathrm{cl}\, N_\varepsilon(Q) \tag{17}$$

for all $(t,x,u) \in \mathbb{R}_0^+ \times M \times \mathcal{U}$. In particular, this implies that Q is also controlled invariant with respect to system (16). Now we define for every $\tau > 0$ the set

$$\mathcal{D}(\tau) := [0,\tau] \times \mathrm{cl}\, N_\varepsilon(Q) \times \mathcal{U}$$

and the number

$$L_\varepsilon(\tau) := \sup_{(t,x,u)\in\mathcal{D}(\tau)} \|D\tilde{\varphi}_{t,u}(x)\|, \quad L_\varepsilon := L_\varepsilon(1). \tag{18}$$

Since $\tilde{\varphi}_{0,u}(x) \equiv x$ on $M \times \mathcal{U}$, we have

$$L_\varepsilon(\tau) \geq \sup_{(x,u)\in \mathrm{cl}\, N_\varepsilon(Q)\times\mathcal{U}} \|D\tilde{\varphi}_{0,u}(x)\| = \sup_{x\in \mathrm{cl}\, N_\varepsilon(Q)} \|\mathrm{id}_{T_xM}\| = 1. \tag{19}$$

Let $\lambda(t,x,u) := \lambda_{\max}(S\nabla\tilde{F}_{u(t)}(\tilde{\varphi}_{t,u}(x)))$ for all $(t,x,u) \in \mathbb{R}_0^+ \times M \times \mathcal{U}$. Then, by the Wazewski Inequality (Proposition 9(ii)), we obtain

$$L_\varepsilon(\tau) \leq \sup_{(t,x,u)\in\mathcal{D}(\tau)} \exp\left(\int_0^t \lambda(s,x,u)ds\right)$$

$$\begin{aligned}
&\leq \sup_{(t,x,u)\in\mathcal{D}(\tau)} \exp\left(\int_0^t \max\{0,\lambda(s,x,u)\}ds\right)\\
&\leq \sup_{(x,u)\in \mathrm{cl}\, N_\varepsilon(Q)\times\mathcal{U}} \exp\left(\int_0^\tau \max\{0,\lambda(s,x,u)\}ds\right)\\
&\leq \sup_{(x,u)\in \mathrm{cl}\, N_\varepsilon(Q)\times\mathcal{U}} \exp\left(\tau \operatorname*{ess\,sup}_{t\in[0,\tau]} \max\{0,\lambda(t,x,u)\}\right)\\
&= \sup_{(x,u)\in \mathrm{cl}\, N_\varepsilon(Q)\times\mathcal{U}} \exp\left(\tau \operatorname*{ess\,sup}_{t\in[0,\tau]} \max\{0,\lambda_{\max}(S\nabla\tilde{F}_{u(t)}(\tilde{\varphi}_{t,u}(x)))\}\right)\\
&\leq \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \exp\left(\tau \max\{0,\lambda_{\max}(S\nabla\tilde{F}_v(z))\}\right).
\end{aligned}$$

By definition of $\tilde{F}$ every solution of system (16) starting in $\mathrm{cl}\, N_\varepsilon(Q)$ stays in $\mathrm{cl}\, N_{2\varepsilon}(Q)$ for all positive times. Hence, $\tilde{\varphi}(\mathcal{D}(\tau)) \subset \mathrm{cl}\, N_{2\varepsilon}(Q)$, which by continuity of $(z,v)\mapsto\lambda_{\max}(S\nabla\tilde{F}_v(z))$ implies

$$\begin{aligned}
L_\varepsilon(\tau) &\leq \sup_{(z,v)\in \mathrm{cl}\, N_{2\varepsilon}(Q)\times U} \exp\left(\tau \max\{0,\lambda_{\max}(S\nabla\tilde{F}_v(z))\}\right)\\
&= \exp\left(\tau\max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_{2\varepsilon}(Q)\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\}\right) < \infty.
\end{aligned}$$

Hence, $L_\varepsilon(\tau)\in[1,\infty)$ for all $\tau>0$. We further obtain

$$\frac{1}{\tau}\ln L_\varepsilon(\tau) \leq \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \max\left\{0,\lambda_{\max}(S\nabla\tilde{F}_v(z))\right\}. \tag{20}$$

<u>Step 2:</u> We show that the following estimate holds:

$$h_{\mathrm{inv}}^+(\varepsilon,K,Q) \leq \ln(L_\varepsilon)\cdot\overline{\dim}_B(K). \tag{21}$$

To this end, first assume that $L_\varepsilon>1$. Let $T>0$ be chosen arbitrarily and let $\mathcal{S}^+=\{(y_1,u_1),\dots,(y_n,u_n)\}$ be a minimal strongly (T,ε,K,Q)-spanning set with respect to system (7). (Note that this implies $n=r_{\mathrm{inv}}^+(T,\varepsilon,K,Q)$.) Then, by (17), $\mathcal{S}^+$ is also minimal strongly (T,ε,K,Q)-spanning with respect to system (16). We define

$$K_j := \left\{x\in M \ :\ \max_{t\in[0,T]} d(\tilde{\varphi}(t,x,u_j),\tilde{\varphi}(t,y_j,u_j))<\varepsilon\right\},\quad j=1,\dots,n.$$

By the definition of strongly (T,ε,K,Q)-spanning sets we have $K\subset\bigcup_{j=1}^n K_j$. Let

$$r(\varepsilon,T) := \varepsilon L_\varepsilon^{-(\lfloor T\rfloor+1)},$$

We want to prove that

$$B_{r(\varepsilon,T)}(y_j)\subset K_j \quad \text{for } j=1,\dots,n.$$

To this end, let $x \in B_{r(\varepsilon,T)}(y_j)$ be chosen arbitrarily for some $j \in \{1,\dots,n\}$, and let $t \in [0,T]$ and $s := t - \lfloor t \rfloor$. By the cocycle property (8) $\tilde{\varphi}_{t,u_j}$ decomposes into $\lfloor t \rfloor + 1$ maps in the following way:

$$\tilde{\varphi}_{t,u_j} = \tilde{\varphi}_{s,\Theta_{\lfloor t \rfloor} u_j} \circ \tilde{\varphi}_{1,\Theta_{\lfloor t \rfloor - 1} u_j} \circ \dots \circ \tilde{\varphi}_{1,\Theta_1 u_j} \circ \tilde{\varphi}_{1,u_j}.$$

Let $c : [0,1] \to M$ be a shortest geodesic joining x and y_j, which exists by the choice of ε. Since $\tilde{\varphi}_{1,u_j} \circ c$ joins $\tilde{\varphi}(1,x,u_j)$ and $\tilde{\varphi}(1,y_j,u_j)$, we get

$$\begin{aligned}
d(\tilde{\varphi}(1,x,u_j),\tilde{\varphi}(1,y_j,u_j)) &\leq \int_0^1 \left\| \tfrac{d}{dr} \tilde{\varphi}_{1,u_j}(c(r)) \right\| dr \\
&= \int_0^1 \left\| D\tilde{\varphi}_{1,u_j}(c(r))\dot{c}(r) \right\| dr \\
&\leq \int_0^1 \left\| D\tilde{\varphi}_{1,u_j}(c(r)) \right\| \|\dot{c}(r)\| dr \\
&\leq \sup_{(z,v) \in \mathrm{cl}\, N_\varepsilon(Q) \times \mathcal{U}} \|D\tilde{\varphi}_{1,v}(z)\| \int_0^1 \|\dot{c}(r)\|\, dr \\
&\leq L_\varepsilon d(x,y_j) < L_\varepsilon r(\varepsilon,T) = \varepsilon L_\varepsilon^{-\lfloor T \rfloor} \leq \varepsilon.
\end{aligned}$$

In the last inequality we used that $L_\varepsilon \geq 1$. Now (if $t \geq 2$) we can choose a shortest geodesic joining $\tilde{\varphi}(1,x,u_j)$ and $\tilde{\varphi}(1,y_j,u_j)$ and estimate the distance of $\tilde{\varphi}(2,x,u_j)$ and $\tilde{\varphi}(2,y_j,u_j)$ in the same way. Recursively, for $l = 1,\dots,\lfloor t \rfloor - 1$ we obtain

$$\begin{aligned}
d(\tilde{\varphi}_{1,\Theta_l u_j} \circ \dots \circ \tilde{\varphi}_{1,u_j}(x), \tilde{\varphi}_{1,\Theta_l u_j} \circ \dots \circ \tilde{\varphi}_{1,u_j}(y_j)) &\leq L_\varepsilon^l d(x,y_j) \\
&< \varepsilon L_\varepsilon^{-\lfloor T \rfloor - 1 + l} \leq \varepsilon,
\end{aligned}$$

and thus also $d(\tilde{\varphi}_{t,u_j}(x), \tilde{\varphi}_{t,u_j}(y_j)) < \varepsilon$. This proves that $B_{r(\varepsilon,T)}(y_j) \subset K_j$. Now assume to the contrary that $N := N(r(\varepsilon,T),K) < r_{\mathrm{inv}}^+(T,\varepsilon,K,Q) = n$, where $N(r(\varepsilon,T),K)$ denotes the minimal number of $r(\varepsilon,T)$-balls necessary to cover the set K. Then K can be covered by N balls of radius $r(\varepsilon,T)$, which can be assumed to be centered at points $z_1,\dots,z_N \in Q$ by Lemma 3. Now we assign to each z_j a control function $v_j \in \mathcal{U}$ such that $(z_j,v_j) \in \mathcal{Q}$, and we define $\tilde{\mathcal{S}}^+ := \{(z_1,v_1),\dots,(z_N,v_N)\}$. Then $\tilde{\mathcal{S}}^+$ is strongly (T,ε,K,Q)-spanning, since for every $x \in K$ we have $x \in B_{r(\varepsilon,T)}(z_j)$ for some $j \in \{1,\dots,N\}$ and we have shown that $d(x,z_j) < r(\varepsilon,T)$ implies $\max_{t\in[0,T]} d(\tilde{\varphi}(t,x,v_j),\tilde{\varphi}(t,z_j,v_j)) < \varepsilon$. Since $\mathcal{S}^+$ is minimal, this is a contradiction. Hence,

$$r_{\mathrm{inv}}^+(T,\varepsilon,K,Q) \leq N(r(\varepsilon,T),K). \tag{22}$$

We have $\ln r(\varepsilon,T) = \ln(\varepsilon L_\varepsilon^{-(\lfloor T \rfloor + 1)}) = \ln(\varepsilon) - (\lfloor T \rfloor + 1)\ln(L_\varepsilon)$ and thus

$$T \geq \lfloor T \rfloor = \frac{\ln(\varepsilon) - \ln(r(\varepsilon,T))}{\ln L_\varepsilon} - 1 = -\frac{\ln r(\varepsilon,T)}{\ln L_\varepsilon}\left(1 + \frac{\ln(L_\varepsilon) - \ln(\varepsilon)}{\ln r(\varepsilon,T)}\right). \tag{23}$$

Note that $\left(1+\frac{\ln(L_\varepsilon)-\ln(\varepsilon)}{\ln r(\varepsilon,T)}\right) \to 1$ for $T \to \infty$. This yields

$$\begin{aligned}
h^+_{\mathrm{inv}}(\varepsilon,K,Q) &= \limsup_{T\to\infty} \frac{\ln r^+_{\mathrm{inv}}(T,\varepsilon,K,Q)}{T} \\
&\overset{(22)}{\leq} \limsup_{T\to\infty} \frac{\ln N(r(\varepsilon,T),K)}{T} \\
&= \ln(L_\varepsilon) \limsup_{T\to\infty} \frac{\ln N(r(\varepsilon,T),K)}{\ln(L_\varepsilon)T} \\
&\overset{(23)}{\leq} \ln(L_\varepsilon) \limsup_{T\to\infty} \frac{\ln N(r(\varepsilon,T),K)}{-\ln r(\varepsilon,T)\left(1+\frac{\ln(L_\varepsilon)-\ln(\varepsilon)}{\ln r(\varepsilon,T)}\right)} \\
&= \ln(L_\varepsilon) \limsup_{T\to\infty} \frac{\ln N(r(\varepsilon,T),K)}{\ln r(\varepsilon,T)^{-1}} \leq \ln(L_\varepsilon)\cdot \overline{\dim}_B(K).
\end{aligned}$$

If $L_\varepsilon = 1$, we can prove the same estimate with $L_\varepsilon + \delta = 1+\delta$ for every $\delta > 0$ and hence, for $\delta \searrow 0$, we obtain $h^+_{\mathrm{inv}}(\varepsilon,K,Q) = 0$.
<u>*Step 3:*</u> We complete the proof. To this end, consider for every $\tau > 0$ the system

$$\dot{x}(t) = \tau \cdot \tilde{F}(x(t),u(t)), \quad u \in \mathcal{U}. \tag{24}$$

Then, by Proposition 7 Q is also controlled invariant with respect to each of these systems, and we obtain for every $\tau > 0$ the estimate

$$h_{\mathrm{inv}}(\varepsilon,K,Q;\tilde{F}) = \tfrac{1}{\tau} h_{\mathrm{inv}}(\varepsilon,K,Q;\tau\tilde{F}) \overset{(9)}{\leq} \tfrac{1}{\tau} h^+_{\mathrm{inv}}(\varepsilon,K,Q;\tau\tilde{F}). \tag{25}$$

Now we apply the estimate (21) to system (24). Denote the cocycle of system (24) by $\tilde{\varphi}^\tau$. Then it is easy to see that

$$\tilde{\varphi}^\tau(\tfrac{t}{\tau},x,\tilde{u}) = \tilde{\varphi}(t,x,u) \quad \text{for all } (t,x,u) \in \mathbb{R}\times M \times \mathcal{U},$$

where $\tilde{u}(t) \equiv u(t\tau)$ (see also the proof of [3, Prop. 3.4(iv)]). Hence,

$$\begin{aligned}
\sup_{(t,x,u)\in\mathcal{D}(1)} \left\| D\tilde{\varphi}^\tau_{t,u}(x)\right\| &= \sup_{(t,x,u)\in\mathcal{D}(1)} \|D\tilde{\varphi}_{t\tau,u}(x)\| \\
&= \sup_{(t,x,u)\in\mathcal{D}(\tau)} \|D\tilde{\varphi}_{t,u}(x)\| = L_\varepsilon(\tau).
\end{aligned}$$

Consequently, from (25) we obtain

$$\begin{aligned}
h_{\mathrm{inv}}(\varepsilon,K,Q) &\leq \tfrac{1}{\tau}\ln(L_\varepsilon(\tau))\cdot\overline{\dim}_B(K) \\
&\overset{(20)}{\leq} \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \max\{0,\lambda_{\max}(S\nabla\tilde{F}_v(z))\}\cdot\overline{\dim}_B(K) \\
&= \max\left\{0, \sup_{(z,v)\in\tilde{\varphi}(\mathcal{D}(\tau))\times U} \lambda_{\max}(S\nabla\tilde{F}_v(z))\right\}\cdot\overline{\dim}_B(K).
\end{aligned}$$

Let $z \in \tilde{\varphi}(D(\tau))$. Then $z = \tilde{\varphi}(t,x,u)$ for some $(t,x,u) \in [0,\tau] \times \operatorname{cl} N_\varepsilon(Q) \times \mathcal{U}$. If u is a piecewise constant control function, then the corresponding trajectory $\tilde{\varphi}(\cdot,x,u)$ is piecewise continuously differentiable, and hence we can measure its length by taking the integral over $\|\frac{d}{dt}\tilde{\varphi}_{x,u}(t)\|$. This implies that for $t \in [0,\tau]$

$$\begin{aligned} d(x,\tilde{\varphi}(t,x,u)) \leq\ & \int_0^t \left\|\tfrac{d}{dt}\tilde{\varphi}_{x,u}(t)\right\| dt \leq \int_0^\tau \left\|\tfrac{d}{dt}\tilde{\varphi}_{x,u}(t)\right\| dt \\ =\ & \int_0^\tau \left\|\tilde{F}(\tilde{\varphi}(t,x,u),u(t))\right\| dt \\ \leq\ & \underbrace{\max_{(z,v)\in \operatorname{cl} N_{2\varepsilon}(Q)\times U} \left\|\tilde{F}(z,v)\right\|}_{=:C} \int_0^\tau dt = C\tau. \end{aligned}$$

The same inequality for arbitrary admissible control functions follows from Proposition 4. This implies

$$\tilde{\varphi}(\mathcal{D}(\tau)) \subset \operatorname{cl} N_{\min\{2\varepsilon,\varepsilon+\tau C\}}(Q) \quad \text{for every } \tau > 0.$$

For $\tau > 0$ with $\varepsilon + \tau C < 2\varepsilon$ we obtain

$$h_{\text{inv}}(\varepsilon,K,Q) \leq \max\left\{0, \max_{(z,v)\in \operatorname{cl} N_{\varepsilon+\tau C}(Q)\times U} \lambda_{\max}(S\nabla \tilde{F}_v(z))\right\} \cdot \overline{\dim}_B(K).$$

Now take a sequence $(\tau_n)_{n\in\mathbb{N}}$, $\tau_n > 0$, with $\tau_n \searrow 0$. Let $(z_n,v_n) \in \operatorname{cl} N_{\varepsilon+\tau_n C}(Q) \times U$ be a point where the maximum above is attained. By compactness we may assume that $(z_n,v_n) \to (z^*,v^*) \in \operatorname{cl} N_\varepsilon(Q) \times U$ for $n \to \infty$. Then

$$\lambda_{\max}(S\nabla \tilde{F}_{v^*}(z^*)) = \max_{(z,v)\in \operatorname{cl} N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla \tilde{F}_v(z)), \tag{26}$$

since otherwise there exists $(z^{**},v^{**}) \in \operatorname{cl} N_\varepsilon(Q) \times U$ with

$$\lambda_{\max}(S\nabla \tilde{F}_{v^{**}}(z^{**})) > \lambda_{\max}(S\nabla \tilde{F}_{v^*}(z^*)),$$

which, by continuity of $(z,v) \mapsto \lambda_{\max}(S\nabla \tilde{F}_v(z))$, implies

$$\begin{aligned} \lambda_{\max}(S\nabla \tilde{F}_{v_n}(z_n)) =\ & \max_{(z,v)\in \operatorname{cl} N_{\varepsilon+\tau_n C}(Q)\times U} \lambda_{\max}(S\nabla \tilde{F}_v(z)) \\ <\ & \lambda_{\max}(S\nabla \tilde{F}_{v^{**}}(z^{**})) \\ \leq\ & \max_{(z,v)\in \operatorname{cl} N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla \tilde{F}_v(z)) \end{aligned}$$

for n large enough. This is a contradiction, since the maximum on $\mathrm{cl}\, N_\varepsilon(Q) \times U$ cannot be greater than the maximum on $\mathrm{cl}\, N_{\varepsilon+\tau_n C}(Q) \times U$. Hence,

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon, K, Q) &\leq \lim_{n\to\infty} \max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_{\varepsilon+\tau_n C}(Q)\times U} \lambda_{\max}(S\nabla \tilde{F}_v(z))\right\} \cdot \overline{\dim}_B(K) \\ &= \lim_{n\to\infty} \max\left\{0, \lambda_{\max}(S\nabla \tilde{F}_{v_n}(z_n))\right\} \cdot \overline{\dim}_B(K) \\ &= \max\left\{0, \lambda_{\max}(S\nabla \tilde{F}_{v^*}(z^*))\right\} \cdot \overline{\dim}_B(K) \\ &\overset{(26)}{=} \max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla \tilde{F}_v(z))\right\} \cdot \overline{\dim}_B(K) \\ &= \max\left\{0, \max_{(z,v)\in \mathrm{cl}\, N_\varepsilon(Q)\times U} \lambda_{\max}(S\nabla F_v(z))\right\} \cdot \overline{\dim}_B(K). \end{aligned}$$

The last equality follows from the fact that $\tilde{F}$ and F coincide on $\mathrm{cl}\, N_\varepsilon(Q)\times U$. With the same arguments it follows that

$$\begin{aligned} h_{\mathrm{inv}}(K, Q) &= \lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon, K, Q) \\ &\leq \max\left\{0, \max_{(z,v)\in Q\times U} \lambda_{\max}(S\nabla F_v(z))\right\} \cdot \overline{\dim}_B(K), \end{aligned}$$

which finishes the proof. □

By considering Riemannian metrics which are conformally equivalent to a given one, we obtain the following corollary.

Corollary 13 *Under the assumptions of Theorem 12 let $W \subset M$ be an open neighborhood of Q and $\alpha : W \to \mathbb{R}$ a smooth function. Then*

$$h_{\mathrm{inv}}(K, Q) \leq \max\left\{0, \max_{(x,u)\in Q\times U} (\lambda_{\max}(S\nabla F_u(x)) + \mathcal{L}_{F_u}\alpha(x))\right\} \cdot \overline{\dim}_B(K).$$

Proof: We define a new Riemannian metric $\tilde{g}$ on W by

$$\tilde{g}(x) := e^{2\alpha(x)} g(x) \quad \text{for all } x \in W$$

and we let $\tilde{\nabla}$ denote the Levi-Civita connection associated with $\tilde{g}$. Then, by (4), for every $f \in \mathcal{X}(M)$ the matrix representation of $S\tilde{\nabla} f$ with respect to a chart (ϕ, V) is given by

$$\begin{aligned} 2\left[S\tilde{\nabla} f\right]_{\mu\nu} &= \frac{\partial f^\mu}{\partial \phi^\nu} + \sum_{\theta,\kappa} \tilde{g}^{\mu\theta} \frac{\partial f^\kappa}{\partial \phi^\theta} \tilde{g}_{\kappa\nu} + \sum_{i,l} f^i \tilde{g}^{\mu l} \frac{\partial \tilde{g}_{\nu l}}{\partial \phi^i} \\ &= \frac{\partial f^\mu}{\partial \phi^\nu} + \sum_{\theta,\kappa} g^{\mu\theta} \frac{\partial f^\kappa}{\partial \phi^\theta} g_{\kappa\nu} + \sum_{i,l} f^i e^{-2\alpha} g^{\mu l} \frac{\partial (e^{2\alpha} g_{\nu l})}{\partial \phi^i} \\ &= \frac{\partial f^\mu}{\partial \phi^\nu} + \sum_{\theta,\kappa} g^{\mu\theta} \frac{\partial f^\kappa}{\partial \phi^\theta} g_{\kappa\nu} \end{aligned}$$

$$
\begin{aligned}
&+ \ e^{-2\alpha} \sum_{i,l} f^i g^{\mu l} \left[e^{2\alpha} \frac{\partial g_{\nu l}}{\partial \phi^i} + 2e^{2\alpha} g_{\nu l} \frac{\partial \alpha}{\partial \phi^i} \right] \\
&= \ [S\nabla f]_{\mu\nu} + 2 \sum_{i,l} f^i g^{\mu l} g_{l\nu} \frac{\partial \alpha}{\partial \phi^i}.
\end{aligned}
$$

Since $\sum_l g^{\mu l} g_{l\nu} = \delta_{\mu\nu}$, we obtain

$$
\left[S\tilde{\nabla} f \right]_{\mu\nu} = [S\nabla f]_{\mu\nu} + \left(\sum_i f^i \frac{\partial \alpha}{\partial \phi^i} \right) \delta_{\mu\nu} = [S\nabla f]_{\mu\nu} + (\mathcal{L}_f \alpha)\delta_{\mu\nu}.
$$

Hence, the assertion follows from Theorem 12. □

The second main result gives a lower bound on $h_{\mathrm{inv}}(K, Q)$ in terms of the divergence of the right-hand side vector fields of the given control system:

Theorem 14 *Consider control system* (7) *and let $K, Q \subset M$ be compact sets with $K \subset Q$ and Q being controlled invariant. Let ω be a smooth volume form on M and assume that $\mu_\omega(K) > 0$. Then the estimate*

$$
h_{\mathrm{inv}}(K, Q) \geq \max \left\{ 0, \min_{(x,u) \in Q \times U} \mathrm{div}_\omega F_u(x) \right\}
$$

holds.

Proof: For arbitrary $T, \varepsilon > 0$ let $\mathcal{S} = \{u_1, \ldots, u_n\}$ be a minimal (T, ε, K, Q)-spanning set and define

$$
K_j := \{x \in K \mid \varphi([0, T], x, u_j) \subset N_\varepsilon(Q)\}, \quad j = 1, \ldots, n.
$$

Then, by definition of (T, ε, K, Q)-spanning sets, $K = \bigcup_{j=1}^n K_j$. For each $j \in \{1, \ldots, n\}$ the set K_j is a Borel set, since it is the intersection of the compact set K and the open set $\{x \in M \mid \varphi([0, T], x, u_j) \subset N_\varepsilon(Q)\}$. The solution map $\varphi_{T,u_j} : M \to M$ is a diffeomorphism and therefore also $\varphi_{T,u_j}(K_j)$ is a Borel set. Hence, we get

$$
\mu_\omega(\varphi_{T,u_j}(K_j)) \leq \mu_\omega(N_\varepsilon(Q)), \quad j = 1, \ldots, n. \tag{27}
$$

For the ω-measure of $\varphi_{T,u_j}(K_j)$ we obtain

$$
\begin{aligned}
\mu_\omega(\varphi_{T,u_j}(K_j)) &= \int_{\varphi_{T,u_j}(K_j)} d\mu_\omega \overset{(6)}{=} \int_{K_j} |\det{}_\omega D\varphi_{T,u_j}(x)| d\mu_\omega(x) \\
&\geq \int_{K_j} d\mu_\omega \inf_{\substack{(x,u) \in K \times \mathcal{U} \\ \varphi([0,T],x,u) \subset N_\varepsilon(Q)}} |\det{}_\omega D\varphi_{T,u}(x)| \\
&= \mu_\omega(K_j) \inf_{\substack{(x,u) \in K \times \mathcal{U} \\ \varphi([0,T],x,u) \subset N_\varepsilon(Q)}} |\det{}_\omega D\varphi_{T,u}(x)| \,.
\end{aligned}
$$

By the Liouville Formula (Proposition 10) this implies

$$\mu_\omega(\varphi_{T,u_j}(K_j)) \geq \mu_\omega(K_j) \cdot \inf_{\substack{(x,u)\in K\times\mathcal{U} \\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} \exp\left(\int_0^T \operatorname{div}_\omega F_{u(s)}(\varphi(s,x,u))ds\right).$$

Let

$$V(\varepsilon,T) := \inf_{\substack{(x,u)\in K\times\mathcal{U} \\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} \exp\left(\int_0^T \operatorname{div}_\omega F_{u(s)}(\varphi(s,x,u))ds\right).$$

We may assume that ε is chosen small enough that $\operatorname{cl} N_\varepsilon(Q)$ is compact. For every $(x,u) \in K \times \mathcal{U}$ with $\varphi([0,T],x,u) \subset N_\varepsilon(Q)$ it holds that

$$\begin{aligned}\exp\left(\int_0^T \operatorname{div}_\omega F_{u(s)}(\varphi(s,x,u))ds\right) &\geq \exp\left(T \min_{(z,u)\in\operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(z)\right) \\ &= \min_{(z,u)\in\operatorname{cl} N_\varepsilon(Q)\times U} \exp\left(T \operatorname{div}_\omega F_u(z)\right),\end{aligned}$$

which implies

$$V(\varepsilon,T) \geq \min_{(z,u)\in\operatorname{cl} N_\varepsilon(Q)\times U} \exp\left(T \operatorname{div}_\omega F_u(z)\right) > 0. \tag{28}$$

We obtain

$$\mu_\omega(K_j) \leq \frac{\mu_\omega(\varphi_{T,u_j}(K_j))}{V(\varepsilon,T)} \overset{(27)}{\leq} \frac{\mu_\omega(N_\varepsilon(Q))}{V(\varepsilon,T)}. \tag{29}$$

Let $j_0 \in \{1,\ldots,n\}$ be chosen such that $\mu_\omega(K_{j_0}) = \max_{j=1,\ldots,n} \mu_\omega(K_j)$. Then

$$\mu_\omega(K) \leq \mu_\omega\Big(\bigcup_{j=1}^n K_j\Big) \leq n \cdot \mu_\omega(K_{j_0}) \overset{(29)}{\leq} n \cdot \frac{\mu_\omega(N_\varepsilon(Q))}{V(\varepsilon,T)}.$$

Since $n = r_{\mathrm{inv}}(T,\varepsilon,K,Q)$, we get

$$r_{\mathrm{inv}}(T,\varepsilon,K,Q) \geq \frac{\mu_\omega(K)}{\mu_\omega(N_\varepsilon(Q))} V(\varepsilon,T) \quad \text{for all } T,\varepsilon > 0$$

and hence

$$\begin{aligned} h_{\mathrm{inv}}(\varepsilon,K,Q) &\geq \limsup_{T\to\infty}\Big[\frac{1}{T}\ln V(\varepsilon,T) + \underbrace{\frac{1}{T}\ln\frac{\mu_\omega(K)}{\mu_\omega(N_\varepsilon(Q))}}_{\to 0}\Big] \\ &\overset{(28)}{\geq} \limsup_{T\to\infty} \min_{(z,u)\in\operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(z) \\ &= \min_{(x,u)\in\operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(x). \end{aligned}$$

For $\varepsilon \searrow 0$ we have $\min_{(x,u)\in\operatorname{cl} N_\varepsilon(Q)\times U} \operatorname{div}_\omega F_u(x) \to \min_{(x,u)\in Q\times U} \operatorname{div}_\omega F_u(x)$, which can be seen as follows: Assume to the contrary that there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there is $(x_n, u_n) \in \mathrm{cl}\, N_{1/n}(Q) \times U$ with

$$\mathrm{div}_\omega\, F_{u_n}(x_n) = \min_{(x,u)\in \mathrm{cl}\, N_{1/n}(Q)\times U} \mathrm{div}_\omega\, F_u(x)$$

and

$$\min_{(x,u)\in Q\times U} F_u(x) - \mathrm{div}_\omega\, F_{u_n}(x_n) \geq \delta.$$

By compactness of $\mathrm{cl}\, N_{1/n}(Q) \times U$ we may assume that (x_n, u_n) converges to some $(x_*, u_*) \in Q \times U$, which, by continuity of $(x,u) \mapsto \mathrm{div}_\omega\, F_u(x)$, leads to the contradiction

$$\mathrm{div}_\omega\, F_{u_*}(x_*) + \delta \leq \min_{(x,u)\in Q\times U} F_u(x) \leq \mathrm{div}_\omega\, F_{u_*}(x_*).$$

Hence, the assertion is true. $\square$

Analogously, as for Theorem 12, we obtain a whole family of bounds if we consider not only one volume form, but all volume forms which are derived from a given one by multiplication with a smooth positive function:

Corollary 15 *Under the assumptions of Theorem 14 let $\alpha : W \to \mathbb{R}$ be a smooth function, defined on an open neighborhood W of Q. Then*

$$h_{\mathrm{inv}}(K,Q) \geq \max\left\{0, \min_{(x,u)\in Q\times U} \left[\mathrm{div}_\omega\, F_u(x) + \mathcal{L}_{F_u}\alpha(x)\right]\right\}. \tag{30}$$

Proof: On W consider the volume form $\omega' := \beta\cdot\omega$ with $\beta(x) \equiv e^{\alpha(x)}$. Using a smooth cut-off function we can extend ω' to M. Then by (5),

$$\begin{aligned}\mathrm{div}_{\omega'}\, F_u(x) &= \mathrm{div}_\omega\, F_u(x) + \frac{\mathcal{L}_{F_u}\beta(x)}{\beta(x)} \\ &= \mathrm{div}_\omega\, F_u(x) + \frac{e^{\alpha(x)}\mathcal{L}_{F_u}\alpha(x)}{e^{\alpha(x)}} = \mathrm{div}_\omega\, F_u(x) + \mathcal{L}_{F_u}\alpha(x).\end{aligned}$$

Now the assertion immediately follows from Theorem 14. $\square$

Remark 16 By Proposition 6, the invariance entropy $h_{\mathrm{inv}}(K,Q)$ is independent of the metric imposed on M. However, the upper bound (1) does depend on the choice of the Riemannian metric g. Hence, one can try to optimize the estimate by taking the infimum over all Riemannian metrics. Analogously, one can try to optimize the lower bound (2) by taking the supremum over all volume forms. We do not know if there is a way to compute these infima and suprema. Corollaries 13 and 15 provide varieties of bounds, which one obtains by considering the conformal class of one particular metric or volume form.

Remark 17 For control systems on Euclidean space, Theorem 4.2 of [3] yields the upper bound

$$h_{\mathrm{inv}}(K,Q) \leq \max_{(x,u)\in Q\times U} \|DF_u(x)\| \cdot \overline{\dim}_B(K), \tag{31}$$

where $\|\cdot\|$ is the operator norm derived from the Euclidean vector norm. In contrast, the "Euclidean version" of our first main result, Theorem 12, gives the estimate

$$h_{\mathrm{inv}}(K,Q) \leq \max\left\{0, \max_{(x,u)\in Q\times U} \lambda_{\max}\left(DF_u(x)^+\right)\right\} \cdot \overline{\dim}_B(K) \tag{32}$$

with $DF_u(x)^+ = \frac{1}{2}[DF_u(x) + DF_u(x)^T]$. Estimate (32) improves (31), since for any matrix $A \in \mathbb{R}^{d\times d}$ the inequality $\lambda_{\max}(\frac{1}{2}(A + A^T)) \leq \|A\|$ holds.

18 Application to Bilinear Systems

Consider a bilinear control system on $\mathbb{R}^{d+1}$, i.e., a system of the form

$$\dot{x}(t) = \left[A_0 + \sum_{i=1}^m u_i(t)A_i\right] x(t), \quad u \in \mathcal{U}, \tag{33}$$

where $A_0, A_1, \ldots, A_m \in \mathbb{R}^{(d+1)\times(d+1)}$. We also use the abbreviation

$$A(u) = A_0 + \sum_{i=1}^m u_i A_i.$$

Any system of this type induces a (nonlinear) control system on the d-dimensional unit sphere

$$S^d = \left\{x \in \mathbb{R}^{d+1} \ : \ \|x\| = 1\right\},$$

given by

$$\dot{s}(t) = (A(u(t)) - s(t)^T A(u(t))s(t)I)s(t), \quad u \in \mathcal{U}, \tag{34}$$

whose solutions are the radial projections of the solutions of (33) (cf. [4, Sec. 7.1]). For the invariance entropy of this system, Theorems 12 and 14 yield the bounds formulated in the following proposition.

Proposition 19 *Consider control sytem* (34). *Let $K, Q \subset S^d$ be compact sets with $K \subset Q$ and Q being controlled invariant. Then*

$$h_{\mathrm{inv}}(K,Q) \leq \max\left\{0, \max_{(s,u)\in Q\times U} \lambda_{\max}(Q_s A(u)^+ - s^T A(u)^+ sI)\right\} \cdot \overline{\dim}_B(K), \tag{35}$$

where $A(u)^+ = \frac{1}{2}(A(u) + A(u)^T)$ *and* Q_s *is the orthogonal projection onto* T_sS^d*. If, in addition,* K *has positive volume, it holds that*

$$h_{\text{inv}}(K, Q) \geq \max\left\{0, \min_{(s,u)\in Q\times U}\left[\operatorname{tr} A(u) - (d+1)\cdot s^T A(u)s\right]\right\}. \qquad (36)$$

Proof: We write

$$G(u, s) = G_u(s) := (A(u) - s^T A(u)sI)s, \quad G : S^d \times \mathbb{R}^m \to TS^d,$$

for the right-hand side of system (34). On S^d we consider the round metric and its induced volume form. In order to compute the upper and lower bounds (1) and (2) for system (34), we first have to determine the covariant derivative of G_u. By [5, Prop. 2.56], $\nabla_v G_u(s)$ is given by the orthogonal projection of $DG_u(s)v$ to $T_sS^d = s^\perp$, where $DG_u(s)$ is the Jacobian of G_u at s, considered as a map from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{d+1}$. An elementary computation gives

$$DG_u(s) = A(u) - s^T A(u)sI - ss^T(A(u) + A(u)^T).$$

With the orthogonal projection $Q_s := I - ss^T$ we obtain

$$\begin{aligned} Q_s DG_u(s) &= (I - ss^T)\left(A(u) - s^T A(u)sI - ss^T(A(u) + A(u)^T)\right) \\ &= (I - ss^T)\left(A(u) - s^T A(u)sI\right). \end{aligned}$$

Hence, $\nabla G_u(s)v = Q_s(A(u) - s^T A(u)sI)v$. For the upper bound (1) we have to compute the symmetrized covariant derivative of G_u. To this end, note that the adjoint $\nabla G_u(s)^*$ of $\nabla G_u(s)$ is the unique linear endomorphism of T_sS^d such that $\langle \nabla G_u(s)v, w\rangle = \langle v, \nabla G_u(s)^* w\rangle$ for all $v, w \in T_sS^d = s^\perp$. Since for $v, w \in s^\perp$ it holds that

$$\begin{aligned} \langle Q_s(A(u) - s^T A(u)sI)v, w\rangle &= \langle \underbrace{v}_{=Q_s v}, (A(u)^T - s^T A(u)sI)\underbrace{Q_s w}_{=w}\rangle \\ &= \langle v, Q_s(A(u)^T - s^T A(u)sI)w\rangle, \end{aligned}$$

we have $\nabla G_u(s)^* v = Q_s(A(u)^T - s^T A(u)sI)v$ and thus

$$\begin{aligned} S\nabla G_u(s) &= \frac{1}{2}\left[Q_s(A(u) - s^T A(u)sI) + Q_s(A(u)^T - s^T A(u)sI)\right] \\ &= \frac{1}{2}Q_s\left[A(u) + A(u)^T - 2s^T A(u)sI\right]. \end{aligned}$$

Writing $A(u)^+$ for $\frac{1}{2}(A(u) + A(u)^T)$ and using that $s^T A(u)s = s^T A(u)^+ s$, we obtain

$$S\nabla G_u(s) = Q_s\left[A(u)^+ - s^T A(u)^+ sI\right] = Q_s A(u)^+ - s^T A(u)^+ sI.$$

Consequently, Theorem 12 yields (35). Now, let $v_1, \ldots, v_d$ be an orthonormal basis of $T_s S^d$. Then the divergence of G_u is given by

$$\begin{aligned}
\operatorname{div} G_u(s) &= \operatorname{tr} \nabla G_u(s) = \sum_{i=1}^{d} \langle Q_s(A(u) - s^T A(u)sI)v_i, v_i \rangle \\
&= \sum_{i=1}^{d} \langle (A(u) - s^T A(u)sI)v_i, Q_s v_i \rangle \\
&= \sum_{i=1}^{d} \langle (A(u) - s^T A(u)sI)v_i, v_i \rangle \\
&= \operatorname{tr}(A(u) - s^T A(u)sI) - \underbrace{\langle (A(u) - s^T A(u)sI)s, s \rangle}_{=0} \\
&= \operatorname{tr} A(u) - (d+1) \cdot s^T A(u)s.
\end{aligned}$$

Hence, Theorem 14 implies (36) in case K has positive volume. □

Remark 20 Finally, we want to remark that existence of compact controlled invariant sets with nonvoid interior (and hence positive volume) for system (34) is guaranteed by [4, Theo. 7.3.3] under the assumption of local accessibility. To be precise, the theorem states the existence of a finite number of control sets with nonvoid interior for the projection of the bilinear system (33) to d-dimensional projective space, which can be viewed as a quotient space of S^d under the equivalence relation which identifies antipodal points. The lifts of these control sets to S^d are controlled invariant with respect to system (34), and—under mild conditions—controlled invariance carries over to their (compact) closures.

21 Acknowledgements

The paper is, essentially, a main part of the author's PhD thesis. I like to thank Prof. Fritz Colonius, my PhD advisor, for many fruitful discussions and for important comments leading to an improvement of this paper.

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